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Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop

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Abstract

In this paper we investigate the operator $H_\beta = -\Delta - \beta\delta(\cdot - \Gamma)$ in $L^2(\mathbb{R}^2)$, where $\beta > 0$ and Γ is a closed C^4 Jordan curve in $\mathbb{R}^2$. We obtain the asymptotic form of each eigenvalue of H_β as β tends to infinity. We also get the asymptotic form of the number of negative eigenvalues of H_β in the strong coupling asymptotic regime. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we study the Schrödinger operator with a δ-interaction on a loop. Let $\Gamma : [0, L] \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2$ be a closed C^4 Jordan curve which is parametrized by the arc length. Let $\gamma : [0, L] \to \mathbb{R}$ be the signed curvature of Γ. For $\beta > 0$, we define

$$q_\beta(f, f) = \|\nabla f\|^2_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma |f(x)|^2 \,\mathrm{d}S, \quad \text{for} \quad f \in H^1(\mathbb{R}^2). \tag{1.1}$$

By H_β we denote the self-adjoint operator associated with the form q_β. The operator H_β is formally written as $-\Delta - \beta\delta(\cdot - \Gamma)$. As the curve is smooth one can alternatively define H_β through boundary conditions expressing the jump of normal derivative across Γ in analogy with the proof of Proposition 2.4. Since Γ is compact in $\mathbb{R}^2$, we have $\sigma_{\mathrm{ess}}(H_\beta) = [0, \infty)$

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by [3] Theorem 3.1. Our main purpose is to study the asymptotic behavior of the negative eigenvalues of H_β as β tends to infinity. We define

$$S = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2(0, L) \tag{1.2}$$

with the domain

$$P = \{\varphi \in H^2(0, L); \quad \varphi(L) = \varphi(0), \quad \varphi'(L) = \varphi'(0)\}. \tag{1.3}$$

For $j \in \mathbb{N}$, we denote by μ_j the jth eigenvalue of S counted with multiplicity. For a finite set A, we denote by $\#A$ the number of the elements of A. Our main results are the following.

Theorem 1. *Let n be an arbitrary integer. There exists $\beta(n) > 0$ such that*

$$\#\sigma_{\mathrm{d}}(H_\beta) \geq n \quad \textit{for} \quad \beta \geq \beta(n).$$

For $\beta \geq \beta(n)$ we denote by $\lambda_n(\beta)$ the nth eigenvalue of H_β counted with multiplicity. Then $\lambda_n(\beta)$ admits an asymptotic expansion of the form

$$\lambda_n(\beta) = -\tfrac{1}{4}\beta^2 + \mu_n + \mathcal{O}(\beta^{-1}\log\beta) \quad \textit{as} \quad \beta \to \infty. \tag{1.4}$$

Theorem 2. *The function $\beta \mapsto \#\sigma_{\mathrm{d}}(H_\beta)$ admits an asymptotic expansion of the form*

$$\#\sigma_{\mathrm{d}}(H_\beta) = \frac{L}{2\pi}\beta + \mathcal{O}(\log\beta) \quad \textit{as} \quad \beta \to \infty. \tag{1.5}$$

The Schrödinger operator with a singular interaction has been studied by numerous authors (see [1–3] and the references therein). The basic concepts of the theory are summarized in the monograph [1]. A particular case of a δ-interaction supported by a curve attracted much less attention (see [3–5,8,10] and a recent paper [6]). In [3] some upper bounds to the number of eigenvalues for a more general class of operators (with β dependent on the arc length parameter) were obtained by the Birman–Schwinger argument (see [3] Theorems 3.4, 3.5 and 4.2). As it is usually the case with the Birman–Schwinger technique, these bounds are sharp for small positive β (see [3] Example 4.1) while they give a poor estimate in the semiclassical regime. On the contrary, our estimate (1.5) is close to optimal for large positive β. Our main tools to prove Theorems 1 and 2 are the Dirichlet–Neumann bracketing and approximate operators with separated variables. We refrain from illustrating the results by solvable examples because these will be given in another work, currently under preparation.

2. Proof of Theorem 1

Let us prepare some quadratic forms and operators which we need in the sequel. For this purpose, we first need the following result.

Lemma 2.1. *Let Φ_a be the map*

$$[0, L) \times (-a, a) \ni (s, u) \mapsto (\Gamma_1(s) - u\Gamma'_2(s), \Gamma_2(s) + u\Gamma'_1(s)) \in \mathbb{R}^2.$$

Then there exists $a_1 > 0$ such that the map Φ_a is injective for any $a \in (0, a_1]$.

Proof. We extend Γ to a periodic function with period L, which we denote by $\tilde{\Gamma}(s) = (\tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s))$. Since Γ is a closed C^4 Jordan curve, we have $\tilde{\Gamma} \in C^4(\mathbb{R})$. We extend γ to a function $\tilde{\gamma}$ on $\mathbb{R}$ by using the formula $\tilde{\gamma}(s) = \tilde{\Gamma}''_1(s)\tilde{\Gamma}'_2(s) - \tilde{\Gamma}''_2(s)\tilde{\Gamma}'_1(s)$. Then $\tilde{\gamma}(\cdot)$ is periodic with period L and $\tilde{\gamma} \in C^2(\mathbb{R})$. By Φ we denote the map

$$\mathbb{R}^2 \ni (s, u) \mapsto (\tilde{\Gamma}_1(s) - u\tilde{\Gamma}'_2(s), \tilde{\Gamma}_2(s) + u\tilde{\Gamma}'_1(s)) \in \mathbb{R}^2.$$

Let $J\Phi$ be the Jacobian matrix of Φ. We put

$$\gamma_+ = \max_{[0,L]} |\gamma(\cdot)|.$$

We have

$$\det J\Phi(s, u) = 1 + u\tilde{\gamma}(s) \geq \frac{1}{2}, \quad \text{for} \quad (s, u) \in \mathbb{R} \times \left[-\frac{1}{2\gamma_+}, \frac{1}{2\gamma_+}\right]. \tag{2.1}$$

In addition, there exists a constant $M > 0$ such that

$$|\partial_y^\alpha \Phi_j(y)| \leq M \quad \text{on} \quad \mathbb{R} \times \left[-\frac{1}{2\gamma_+}, \frac{1}{2\gamma_+}\right] \tag{2.2}$$

for any $1 \leq |\alpha| \leq 2$ and $j = 1, 2$, where $y = (s, u)$ and $\Phi(y) = (\Phi_1(y), \Phi_2(y))$. Combining [11] Lemma 3.6 with (2.1) and (2.2), we claim that there exists $a_0 \in (0, 1/2\gamma_+)$ such that Φ is injective on $[k - a_0, k + a_0] \times [-a_0, a_0]$ for all $k \in \mathbb{R}$. We put

$$\tau = \min_{p \in [a_0, L/2]} \min_{t \in [0,L]} |\tilde{\Gamma}(t) - \tilde{\Gamma}(t + p)|. \tag{2.3}$$

Since $\tilde{\Gamma}$ is injective on $[0, L)$ and $\tilde{\Gamma}(\cdot)$ has period L, we have $\tau > 0$. Put $a_1 = \min\{a_0, \tau/4\}$. Let us show that Φ is injective on $[0, L) \times (-a_1, a_1)$. We first prove the following claim.

(i) Assume that $\Phi(s_1, u_1) = \Phi(s_2, u_2)$, $|s_1 - s_2| \leq L/2$, and $(s_1, u_1), (s_2, u_2) \in \mathbb{R} \times (-a_1, a_1)$. Then we have $(s_1, u_1) = (s_2, u_2)$.

Since $\Phi(s_1, u_1) = \Phi(s_2, u_2)$ and $|\tilde{\Gamma}'_j(\cdot)| \leq 1$ on $\mathbb{R}$ for $j = 1, 2$, we obtain

$$|\tilde{\Gamma}_1(s_1) - \tilde{\Gamma}_1(s_2)| = |u_1\tilde{\Gamma}'_2(s_1) - u_2\tilde{\Gamma}'_2(s_2)| \leq 2a_1,$$

$$|\tilde{\Gamma}_2(s_1) - \tilde{\Gamma}_2(s_2)| = |u_1\tilde{\Gamma}'_1(s_1) - u_2\tilde{\Gamma}'_1(s_2)| \leq 2a_1.$$

So we have $|\tilde{\Gamma}(s_1) - \tilde{\Gamma}(s_2)| \leq 2\sqrt{2}a_1$, and therefore

$$|\tilde{\Gamma}(s_1) - \tilde{\Gamma}(s_2)| < \tau.$$

This together with (2.3) implies that $|s_1 - s_2| < a_0$. Since Φ is injective on $[s_1 - a_0, s_1 + a_0] \times [-a_0, a_0]$ and $\Phi(s_1, u_1) = \Phi(s_2, u_2)$, we get $(s_1, u_1) = (s_2, u_2)$. In this way we proved (i).

Next we shall prove the following implication.

(ii) Assume that $\Phi(s_1, u_1) = \Phi(s_2, u_2)$, $s_1 \leq s_2$, and $(s_1, u_1), (s_2, u_2) \in [0, L) \times (-a_1, a_1)$. Then we have $s_2 - s_1 \leq L/2$.

We prove this by contradiction. Assume that $s_2 - s_1 > L/2$. We put $s_3 = s_2 - L$. Then we get $0 < s_1 - s_3 < L/2$ and $\Phi(s_3, u_2) = \Phi(s_1, u_1)$. As in the proof of (i) we obtain $(s_1, u_1) = (s_3, u_2)$ which violates the fact that $0 < s_1 - s_3 < L/2$, so we proved (ii).

Combining (i) with (ii), we conclude that Φ is injective on $[0, L) \times (-a_1, a_1)$. □

Let $0 < a < a_1$. Let Σ_a be the strip of width $2a$ enclosing Γ:

$$\Sigma_a = \Phi([0, L) \times (-a, a)).$$

Then $\mathbb{R}^2 \backslash \Sigma_a$ consists of two connected components which we denote by Λ_a^{in} and Λ_a^{out}, where Λ_a^{in} is compact. We define

$$q_{a,\beta}^{+}(f, f) = \|\nabla f\|_{L^2(\Sigma_a)}^2 - \beta \int_\Gamma |f(x)|^2\, \mathrm{d}S, \quad \text{for} \quad f \in H_0^1(\Sigma_a),$$

$$q_{a,\beta}^{-}(f, f) = \|\nabla f\|_{L^2(\Sigma_a)}^2 - \beta \int_\Gamma |f(x)|^2\, \mathrm{d}S, \quad \text{for} \quad f \in H^1(\Sigma_a).$$

Let $L_{a,\beta}^{+}$ and $L_{a,\beta}^{-}$ be the self-adjoint operators associated with the forms $q_{a,\beta}^{+}$ and $q_{a,\beta}^{-}$, respectively. By using the Dirichlet–Neumann bracketing (see [9] XIII.15, Proposition 4), we obtain

$$(-\Delta_{\Lambda_a^{\text{in}}}^{\text{N}}) \oplus L_{a,\beta}^{-} \oplus (-\Delta_{\Lambda_a^{\text{out}}}^{\text{N}}) \leq H_\beta \leq (-\Delta_{\Lambda_a^{\text{in}}}^{\text{D}}) \oplus L_{a,\beta}^{+} \oplus (-\Delta_{\Lambda_a^{\text{out}}}^{\text{D}}) \tag{2.4}$$

in $L^2(\Lambda_a^{\text{in}}) \oplus L^2(\Sigma_a) \oplus L^2(\Lambda_a^{\text{out}})$. In order to estimate the negative eigenvalues of H_β, it is sufficient to estimate those of $L_{a,\beta}^{+}$ and $L_{a,\beta}^{-}$, because the other operators involved in (2.4) are positive.

To this aim we introduce two operators in $L^2((0, L) \times (-a, a))$ which are unitarily equivalent to $L_{a,\beta}^{+}$ and $L_{a,\beta}^{-}$, respectively. We define

$$Q_a^{+} = \{\varphi \in H^1((0, L) \times (-a, a)); \quad \varphi(L, \cdot) = \varphi(0, \cdot) \quad \text{on} \quad (-a, a),$$
$$\varphi(\cdot, a) = \varphi(\cdot, -a) = 0 \quad \text{on} \quad (0, L)\},$$

$$Q_a^{-} = \{\varphi \in H^1((0, L) \times (-a, a)); \quad \varphi(L, \cdot) = \varphi(0, \cdot) \quad \text{on} \quad (-a, a)\},$$

$$\begin{aligned} b_{a,\beta}^{+}(f, f) = &\int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} \left|\frac{\partial f}{\partial s}\right|^2 \mathrm{d}u\, \mathrm{d}s + \int_0^L \int_{-a}^a \left|\frac{\partial f}{\partial u}\right|^2 \mathrm{d}u\, \mathrm{d}s \\ &+ \int_0^L \int_{-a}^a V(s, u)|f|^2\, \mathrm{d}s\, \mathrm{d}u - \beta \int_0^L |f(s, 0)|^2\, \mathrm{d}s, \quad \text{for} \quad f \in Q_a^{+}, \end{aligned}$$

$$\begin{aligned} b_{a,\beta}^{-}(f, f) = &\int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} \left|\frac{\partial f}{\partial s}\right|^2 \mathrm{d}u\, \mathrm{d}s + \int_0^L \int_{-a}^a \left|\frac{\partial f}{\partial u}\right|^2 \mathrm{d}u\, \mathrm{d}s \\ &+ \int_0^L \int_{-a}^a V(s, u)|f|^2\, \mathrm{d}s\, \mathrm{d}u - \beta \int_0^L |f(s, 0)|^2\, \mathrm{d}s \\ &- \frac{1}{2} \int_0^L \frac{\gamma(s)}{1 + a\gamma(s)} |f(s, a)|^2\, \mathrm{d}s + \frac{1}{2} \int_0^L \frac{\gamma(s)}{1 - a\gamma(s)} |f(s, -a)|^2\, \mathrm{d}s \end{aligned}$$

for $f \in Q_a^-$, where

$$V(s,u) = \tfrac{1}{2}(1+u\gamma(s))^{-3}u\gamma''(s) - \tfrac{5}{4}(1+u\gamma(s))^{-4}u^2\gamma'(s)^2 \\ -\tfrac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2.$$

Let $B_{a,\beta}^+$ and $B_{a,\beta}^-$ be the self-adjoint operators associated with the forms $b_{a,\beta}^+$ and $b_{a,\beta}^-$, respectively. Then we have the following result.

Lemma 2.2. *The operators $B_{a,\beta}^+$ and $B_{a,\beta}^-$ are unitarily equivalent to $L_{a,\beta}^+$ and $L_{a,\beta}^-$, respectively*.

Proof. We prove the assertion only for $B_{a,\beta}^-$ because that for $B_{a,\beta}^+$ is similar. Given $f \in L^2(\Sigma_a)$, we define

$$(U_a f)(s,u) = (1+u\gamma(s))^{1/2} f(\Phi_a(s,u)), \quad (s,u) \in (0,L)\times(-a,a). \tag{2.5}$$

From Lemma 2.1, we infer that U_a is a unitary operator from $L^2(\Sigma_a)$ to $L^2((0,L)\times(-a,a))$. Since Γ is a closed C^4 Jordan curve, U_a is a bijection from $H^1(\Sigma_a)$ to Q_a^-. Using an integration by parts, we obtain

$$q_{a,\beta}^-(f,g) - b_{a,\beta}^-(U_a f, U_a g) \\ = -\frac{1}{2}\int_{-a}^{a}\left[(1+u\gamma(s))^{-3}\gamma'(s)(U_a f)(s,u)\overline{(U_a g)(s,u)}\right]_{s=0}^{s=L} \mathrm{d}u.$$

Since $U_a f$ and $U_a g$ as elements of Q_a^- satisfy the periodicity condition, we get

$$q_{a,\beta}^-(f,g) = b_{a,\beta}^-(U_a f, U_a g), \quad \text{for} \quad f,\ g \in H^1(\Sigma_a).$$

This together with the first representation theorem (see [7] Theorem VI.2.1) implies that

$$U_a^* B_{a,\beta}^- U_a = L_{a,\beta}^-.$$

This completes the proof of the lemma. □

Next we estimate $B_{a,\beta}^+$ and $B_{a,\beta}^-$ by operators with separated variables. We put

$$\gamma'_+ = \max_{[0,L]}|\gamma'(\cdot)|, \qquad \gamma''_+ = \max_{[0,L]}|\gamma''(\cdot)|,$$

$$V_+(s) = \tfrac{1}{2}(1-a\gamma_+)^{-3}a\gamma''_+ - \tfrac{5}{4}(1+a\gamma_+)^{-4}a^2(\gamma'_+)^2 - \tfrac{1}{4}(1+a\gamma_+)^{-2}\gamma(s)^2,$$

$$V_-(s) = -\tfrac{1}{2}(1-a\gamma_+)^{-3}a\gamma''_+ - \tfrac{5}{4}(1-a\gamma_+)^{-4}a^2(\gamma'_+)^2 - \tfrac{1}{4}(1-a\gamma_+)^{-2}\gamma(s)^2.$$

If $0 < a < (1/2)\gamma_+$, we can define

$$\tilde{b}_{a,\beta}^+(f,f) = (1-a\gamma_+)^{-2}\int_0^L\int_{-a}^{a}\left|\frac{\partial f}{\partial s}\right|^2 \mathrm{d}u\,\mathrm{d}s + \int_0^L\int_{-a}^{a}\left|\frac{\partial f}{\partial u}\right|^2 \mathrm{d}u\,\mathrm{d}s \\ + \int_0^L\int_{-a}^{a} V_+(s)|f|^2\,\mathrm{d}u\,\mathrm{d}s - \beta\int_0^L |f(s,0)|^2\,\mathrm{d}s, \quad \text{for} \quad f \in Q_a^+,$$

$$
\begin{aligned}
\tilde{b}^-_{a,\beta}(f,f) &= (1+a\gamma_+)^{-2}\int_0^L\int_{-a}^a\left|\frac{\partial f}{\partial s}\right|^2\,\mathrm{d}u\,\mathrm{d}s + \int_0^L\int_{-a}^a\left|\frac{\partial f}{\partial u}\right|^2\,\mathrm{d}u\,\mathrm{d}s\\
&\quad + \int_0^L\int_{-a}^a V_-(s)|f|^2\,\mathrm{d}u\,\mathrm{d}s - \beta\int_0^L|f(s,0)|^2\,\mathrm{d}s\\
&\quad -\gamma_+\int_0^L(|f(s,a)|^2+|f(s,-a)|^2)\,\mathrm{d}s, \quad \text{for} \quad f\in Q_a^-.
\end{aligned}
$$

Then we have

$$
b^+_{a,\beta}(f,f) \le \tilde{b}^+_{a,\beta}(f,f), \quad \text{for} \quad f\in Q_a^+, \tag{2.6}
$$

$$
\tilde{b}^-_{a,\beta}(f,f) \le b^-_{a,\beta}(f,f), \quad \text{for} \quad f\in Q_a^-. \tag{2.7}
$$

Let $\tilde{H}^+_{a,\beta}$ and $\tilde{H}^-_{a,\beta}$ be the self-adjoint operators associated with the forms $\tilde{b}^+_{a,\beta}$ and $\tilde{b}^-_{a,\beta}$, respectively. Let $T^+_{a,\beta}$ be the self-adjoint operator associated with the form

$$
t^+_{a,\beta}(f,f) = \int_{-a}^a |f'(u)|^2\,\mathrm{d}u - \beta|f(0)|^2, \quad f\in H_0^1(-a,a).
$$

Let finally $T^-_{a,\beta}$ be the self-adjoint operator associated with the form

$$
t^-_{a,\beta}(f,f) = \int_{-a}^a |f'(u)|^2\,\mathrm{d}u - \beta|f(0)|^2 - \gamma_+(|f(a)|^2+|f(-a)|^2),
$$
$$
f\in H^1(-a,a).
$$

We define

$$
U_a^+ = -(1-a\gamma_+)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_+(s) \quad \text{in} \quad L^2(0,L) \quad \text{with the domain } P,
$$

$$
U_a^- = -(1+a\gamma_+)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_-(s) \quad \text{in} \quad L^2(0,L) \quad \text{with the domain } P.
$$

Then we have

$$
\tilde{H}^+_{a,\beta} = U_a^+\otimes 1 + 1\otimes T^+_{a,\beta}, \qquad \tilde{H}^-_{a,\beta} = U_a^-\otimes 1 + 1\otimes T^-_{a,\beta}. \tag{2.8}
$$

Next we consider the asymptotic behavior of each eigenvalue of $U_a^\pm$ as a tends to zero. Let $\mu_j^\pm(a)$ be the jth eigenvalue of $U_a^\pm$ counted with multiplicity. The following proposition is needed to prove Theorem 2 as well as Theorem 1.

Proposition 2.3. *There exists* $C_1>0$ *such that*

$$
|\mu_j^+(a)-\mu_j| \le C_1 a j^2 \tag{2.9}
$$

and

$$
|\mu_j^-(a)-\mu_j| \le C_1 a j^2 \tag{2.10}
$$

for $j\in\mathbb{N}$ *and* $0<a<1/(2\gamma_+)$, *where* C_1 *is independent of* j, a.

Proof. We define

$$S_0 = -\frac{d^2}{ds^2} \quad \text{in} \quad L^2(0, L) \quad \text{with the domain } P.$$

Notice that the jth eigenvalue of S_0 counted with multiplicity is $4[j/2]^2(\pi/L)^2$. Since

$$\|S - S_0\|_{\mathcal{B}(L^2(0,L))} \leq \tfrac{1}{4}\gamma_+^2,$$

the min–max principle (see [9] Theorem XIII.2) implies that

$$\left|\mu_j - 4\left[\frac{j}{2}\right]^2\left(\frac{\pi}{L}\right)^2\right| \leq \frac{1}{4}\gamma_+^2, \quad \text{for} \quad j \in \mathbb{N}. \tag{2.11}$$

Since

$$U_a^+ - (1 - a\gamma_+)^{-2}S = \tfrac{1}{2}(1 - a\gamma_+)^{-3}a\gamma''_+ - \tfrac{5}{4}(1 + a\gamma_+)^{-4}a^2(\gamma'_+)^2$$
$$+ a\gamma_+(1 + a\gamma_+)^{-2}(1 - a\gamma_+)^{-2}\gamma(s)^2,$$

we infer that there exists $C_0 > 0$ such that

$$\|U_a^+ - (1 - a\gamma_+)^{-2}S\|_{\mathcal{B}(L^2(0,L))} \leq C_0 a, \quad \text{for} \quad 0 < a < 1/(2\gamma_+).$$

This together with the min–max principle implies that

$$|\mu_j^+(a) - (1 - a\gamma_+)^{-2}\mu_j| \leq C_0 a \quad \text{for} \quad 0 < a < 1/(2\gamma_+).$$

Hence we get

$$|\mu_j^+(a) - \mu_j| \leq C_0 a + \frac{a\gamma_+(2 - a\gamma_+)}{(1 - a\gamma_+)^2}|\mu_j|.$$

Combining this with (2.11) we arrive at (2.9).

The proof of (2.10) is similar. □

Next we estimate the first eigenvalue of $T_{a,\beta}^+$.

Proposition 2.4. *Assume that $\beta a > 8/3$. Then $T_{a,\beta}^+$ has only one negative eigenvalue which we denote by $\zeta_{a,\beta}^+$. It satisfies the inequalities*

$$-\tfrac{1}{4}\beta^2 < \zeta_{a,\beta}^+ < -\tfrac{1}{4}\beta^2 + 2\beta^2 \exp(-\tfrac{1}{2}(\beta a)).$$

Proof. Let $k > 0$. We will show that $-k^2$ is an eigenvalue of $T_{a,\beta}^+$ if and only if

$$g_{a,\beta}(k) := \log(\beta - 2k) - \log(\beta + 2k) + 2ka = 0.$$

Assume that $-k^2$ is an eigenvalue of $T_{a,\beta}^+$. Notice that

$$\mathcal{D}(T_{a,\beta}^+) = \{\varphi \in H_0^1(-a, a); \quad \varphi|_{(0,a)} \in H^2(0, a), \quad \varphi|_{(-a,0)} \in H^2(-a, 0),$$
$$\varphi'(+0) - \varphi'(-0) = -\beta\varphi(0)\}.$$

Let a non-zero ψ be the eigenfunction of $T^+_{a,\beta}$ associated with the eigenvalue $-k^2$, then we have

(i) $-\psi''(u) = -k^2\psi(u)$ on $(-a, 0) \cup (0, a)$;
(ii) $\psi(\pm a) = 0$;
(iii) $\psi'(+0) - \psi'(-0) = -\beta\psi(0)$.

Since $T^+_{a,\beta}$ commutes with the parity operator $f(x) \mapsto f(-x)$, the ground state ψ satisfies $\psi(u) = \psi(-u)$ on $[0, a]$. Combining this with (i), we infer that ψ is of the form

$$\psi(u) = \begin{cases} C_1 \mathrm{e}^{ku} + C_2 \mathrm{e}^{-ku}, & u \in (0, a), \\ C_2 \mathrm{e}^{ku} + C_1 \mathrm{e}^{-ku}, & u \in (-a, 0). \end{cases} \tag{2.12}$$

Note that (ii) is equivalent to

$$C_2 = -C_1 \mathrm{e}^{2ka}.$$

In addition, (iii) is equivalent to

$$(2k + \beta)C_1 - (2k - \beta)C_2 = 0.$$

Thus the equation for C_1 and C_2 becomes

$$\begin{pmatrix} 2k + \beta & -(2k - \beta) \\ \mathrm{e}^{2ka} & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{2.13}$$

Since $(C_1, C_2) \neq (0, 0)$, we get

$$\det \begin{pmatrix} 2k + \beta & -(2k - \beta) \\ \mathrm{e}^{2ka} & 1 \end{pmatrix} = 0$$

which is equivalent to $g_{a,\beta}(k) = 0$.

To check the converse, assume that $g_{a,\beta}(k) = 0$. Then (2.13) has a solution $(C_1, C_2) \neq (0, 0)$. It is easy to see that the function ψ from (2.12) satisfies (i)–(iii) and $\psi \in \mathcal{D}(T^+_{a,\beta})$.

Let us show that $g_{a,\beta}(\cdot)$ has a unique zero in $(0, \beta/4)$. We have $g_{a,\beta}(0) = 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}k} g_{a,\beta}(k) = \frac{-4\beta}{\beta^2 - 4k^2} + 2a,$$

we claim that $g_{a,\beta}(\cdot)$ is monotone increasing on $(0, \frac{1}{2}\sqrt{\beta^2 - 2\beta/a})$ and is monotone decreasing on $(\frac{1}{2}\sqrt{\beta^2 - 2\beta/a}, \frac{1}{2}\beta)$. Moreover, we have

$$\lim_{k \to \beta/2 - 0} g_{a,\beta}(k) = -\infty.$$

Hence the function $g_{a,\beta}(\cdot)$ has a unique zero in $(0, \beta/2)$. Since $a\beta > 8/3$, we have $\frac{1}{2}\sqrt{\beta^2 - 2\beta/a} \geq \beta/4$. Consequently, the solution k has the form $k = \beta/2 - s$, $0 < s \leq \beta/4$. Taking into account the relation $g_{a,\beta}(k) = 0$, we get

$$\log 2s = \log(2\beta - s) - \beta a + 2as < \log 2\beta - \tfrac{1}{2}a\beta.$$

So we obtain $s < \beta \exp(-\frac{1}{2}a\beta)$. This completes the proof of Proposition 2.4. □

Next we estimate the first eigenvalue of $T^-_{a,\beta}$.

Proposition 2.5. *Let $a\beta > 8$ and $\beta > \frac{8}{3}\gamma_+$. Then $T^-_{a,\beta}$ has a unique negative eigenvalue $\zeta^-_{a,\beta}$, and moreover we have*

$$-\frac{1}{4}\beta^2 - \frac{2205}{16}\beta^2 \exp\left(-\frac{1}{2}\beta a\right) < \zeta^-_{a,\beta} < -\frac{1}{4}\beta^2.$$

Proof. Let us first show that $T^-_{a,\beta}$ has a unique negative eigenvalue. Let $k > 0$. As in the proof of Proposition 2.4, we infer that $-k^2$ is an eigenvalue of $T^-_{a,\beta}$ if and only if

$$\frac{k\mathrm{e}^{ka} - \gamma_+}{k\mathrm{e}^{-ka} + \gamma_+} = \frac{2k+\beta}{2k-\beta}. \tag{2.14}$$

Since the left side of (2.14) is positive for $k \geq \gamma_+$ and the right side of (2.14) is negative for $0 < k < \beta/2$, (2.14) has no solution in $[\gamma_+, \beta/2)$. We put

$$g(k) = \frac{k\mathrm{e}^{ka} - \gamma_+}{k\mathrm{e}^{-ka} + \gamma_+} \quad \text{and} \quad h(k) = \frac{2k+\beta}{2k-\beta}.$$

Then we get $\lim_{k\to\infty} g(k) = \infty$ and

$$g'(k) = \frac{\gamma_+(\mathrm{e}^{ka} - \mathrm{e}^{-ka}) + 2k^2 a + ka\gamma_+(\mathrm{e}^{ka} + \mathrm{e}^{-ka})}{(k\mathrm{e}^{-ka} + \gamma_+)^2} > 0 \quad \text{for} \quad k > 0.$$

Thus $g(k)$ is monotone increasing on $(0, \infty)$. On the other hand, $h(k)$ is monotone decreasing on $(\beta/2, \infty)$,

$$\lim_{k\to\beta/2+0} h(k) = \infty, \qquad \lim_{k\to\infty} h(k) = 1.$$

Hence (2.14) has a unique solution in $(\beta/2, \infty)$. Since $h(k)$ is monotone decreasing on $(0, \beta/2)$ and $g(0) = h(0)$, we claim that (2.14) has no solution in $(0, \beta/2)$.

Next we show that $g(k) > (2k+\beta)/(2k-\beta)$, for $k \geq \frac{3}{4}\beta$. We have $(2k+\beta)/(2k-\beta) \leq 5$, for $k \geq (3/4)\beta$. For $k \geq (3/4)\beta$, we get

$$g(k) \geq g\left(\frac{3}{4}\beta\right) = \frac{(3/4)\beta \exp((3/4)a\beta) - \gamma_+}{(3/4)\beta \exp(-(3/4)a\beta) + \gamma_+}$$

since $\gamma_+ < \frac{3}{8}\beta < \frac{3}{8}\beta \exp(\frac{3}{4}a\beta)$

$$\geq \frac{(3/8)\beta \exp((3/4)a\beta)}{(3/4)\beta \exp(-(3/4)a\beta) + (3/8)\beta} = \frac{\exp((3/4)a\beta)}{2\exp(-(3/4)a\beta) + 1}$$

since $a\beta > 8$

$$\geq \frac{\mathrm{e}^6}{2\mathrm{e}^{-6} + 1} > 5.$$

So (2.14) has no solution in $[\frac{3}{4}\beta, \infty)$. Hence, the solution k of (2.14) is of the form $k = \beta/2 + s$, $0 < s < \frac{1}{4}\beta$. From (2.14), we get

$$\frac{5\beta}{4s} \geq \frac{2k+\beta}{2k-\beta} = \frac{k\mathrm{e}^{ka} - \gamma_+}{k\mathrm{e}^{-ka} + \gamma_+}$$

since $\gamma_+ < (\frac{3}{8})\beta < (\frac{3}{8})\beta\exp((\frac{1}{2})\beta a)$ and $k\mathrm{e}^{ka} \geq \frac{1}{2}\beta\exp(\frac{1}{2}\beta a)$

$$\geq \frac{(1/8)\beta\exp((1/2)\beta a)}{k\mathrm{e}^{-ka} + \gamma_+}$$

since $k\mathrm{e}^{-ka} < k < (\frac{3}{4})\beta$ and $\gamma_+ < \frac{3}{8}\beta$

$$\geq \frac{(1/8)\beta\exp((1/2)\beta a)}{(9/8)\beta} = \frac{1}{9}\exp\left(\frac{1}{2}\beta a\right).$$

Thus we get $s \leq \frac{45}{4}\beta\exp(-\frac{1}{2}\beta a)$, which gives $k^2 \geq \beta^2/4$ and

$$\begin{aligned} k^2 &= \frac{\beta^2}{4} + \beta s + s^2 \leq \frac{\beta^2}{4} + \frac{45}{4}\beta^2\exp\left(-\frac{1}{2}\beta a\right) + \left(\frac{45}{4}\right)^2\beta^2\exp(-\beta a) \\ &\leq \frac{\beta^2}{4} + \frac{45}{4}\beta^2\exp\left(-\frac{1}{2}\beta a\right) + \left(\frac{45}{4}\right)^2\beta^2\exp\left(-\frac{1}{2}\beta a\right) \\ &= \frac{\beta^2}{4} + \frac{2205}{16}\exp\left(-\frac{1}{2}\beta a\right). \end{aligned}$$

This completes the proof of Proposition 2.5. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We put $a(\beta) = 6\beta^{-1}\log\beta$. Let $\xi^{\pm}_{\beta,j}$ be the jth eigenvalue of $T^{\pm}_{a(\beta),\beta}$. From Propositions 2.4 and 2.5, we have

$$\xi^{\pm}_{\beta,1} = \zeta^{\pm}_{a(\beta),\beta} \quad \text{and} \quad \xi^{\pm}_{\beta,2} \geq 0.$$

From (2.8), we infer that $\{\xi^{\pm}_{\beta,j} + \mu^{\pm}_k(a(\beta))\}_{j,k\in\mathbb{N}}$ is a sequence of all eigenvalues of $\tilde{H}^{\pm}_{a(\beta),\beta}$ counted with multiplicity. From Proposition 2.3, we have

$$\xi^{\pm}_{\beta,j} + \mu^{\pm}_k(a(\beta)) \geq \mu^{\pm}_1(a(\beta)) = \mu_1 + \mathcal{O}(\beta^{-1}\log\beta) \tag{2.15}$$

for $j \geq 2$ and $k \geq 1$. For $j \in \mathbb{N}$, we define

$$\tau^{\pm}_{\beta,j} = \zeta^{\pm}_{a(\beta),\beta} + \mu^{\pm}_j(a(\beta)). \tag{2.16}$$

From Propositions 2.3–2.5, we get

$$\tau^{\pm}_{\beta,j} = -\frac{1}{4}\beta^2 + \mu_j + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty. \tag{2.17}$$

Let $n \in \mathbb{N}$. Combining (2.15) with (2.17), we claim that there exists $\beta(n) > 0$ such that

$$\tau^{+}_{\beta,n} < 0, \qquad \tau^{+}_{\beta,n} < \xi^{+}_{\beta,j} + \mu^{+}_k(a(\beta)), \qquad \tau^{-}_{\beta,n} < \xi^{-}_{\beta,j} + \mu^{-}_k(a(\beta))$$

for $\beta \geq \beta(n)$, $j \geq 2$, and $k \geq 1$. Hence the jth eigenvalue of $\tilde{H}^{\pm}_{a(\beta),\beta}$ counted with multiplicity is $\tau^{\pm}_{\beta,j}$ for $j \leq n$ and $\beta \geq \beta(n)$. Let $\beta \geq \beta(n)$ and denote by $\kappa^{\pm}_j(\beta)$ the jth eigenvalue of $L^{\pm}_{a(\beta),\beta}$. From (2.4) and (2.6), and the min–max principle we obtain

$$\tau^{-}_{\beta,j} \leq \kappa^{-}_j(\beta) \quad \text{and} \quad \kappa^{+}_j(\beta) \leq \tau^{+}_{\beta,j}, \quad \text{for} \quad 1 \leq j \leq n, \tag{2.18}$$

so we have $\kappa^{+}_n(\beta) < 0$. Hence the min–max principle and (2.4) imply that H_β has at least n eigenvalues in $(-\infty, \kappa^{+}_n(\beta))$. For $1 \leq j \leq n$, we denote by $\lambda_j(\beta)$ the jth eigenvalue of H_β. We have

$$\kappa^{-}_j(\beta) \leq \lambda_j(\beta) \leq \kappa^{+}_j(\beta), \quad \text{for} \quad 1 \leq j \leq n.$$

This together with (2.17) and (2.18) implies that

$$\lambda_j(\beta) = -\tfrac{1}{4}\beta^2 + \mu_j + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty, \quad \text{for} \quad 1 \leq j \leq n.$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2

For a self-adjoint operator A, we define

$$N^{-}(A) = \#\{\sigma_{\mathrm{d}}(A) \cap (-\infty, 0)\}.$$

From (2.4), we have $N^{-}(L^{-}_{a,\beta}) \geq \#\sigma_{\mathrm{d}}(H_\beta) \geq N^{-}(L^{+}_{a,\beta})$. On the other hand, Lemma 2.2, (2.6) and (2.7) imply that $N^{-}(\tilde{H}^{-}_{a,\beta}) \geq N^{-}(L^{-}_{a,\beta})$ and $N^{-}(L^{+}_{a,\beta}) \geq N^{-}(\tilde{H}^{+}_{a,\beta})$. In this way we get

$$N^{-}(\tilde{H}^{+}_{a,\beta}) \leq \#\sigma_{\mathrm{d}}(H_\beta) \leq N^{-}(\tilde{H}^{-}_{a,\beta}). \tag{3.1}$$

Recall the relation (2.16). We define

$$K^{\pm}_\beta = \{j \in \mathbb{N}; \quad \tau^{\pm}_{\beta,j} < 0\}$$

and use the following proposition to estimate $N^{-}(\tilde{H}^{\pm}_{a,\beta})$.

Proposition 3.1. *We have*

$$\#K^{\pm}_\beta = \frac{L}{2\pi}\beta + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta \to \infty.$$

Proof. We choose $C_2 > 0$ such that $-(1/4)C_2^2 \leq -1 - (1/4)\gamma_+^2$. Let $\beta \geq \max\{2, C_2\}$. Then we have $(1/4)(\beta - C_2)^2 < (1/4)\beta^2 - 1 - (1/4)\gamma_+^2$. We get

$$K^{+}_\beta = \{j \in \mathbb{N}; \quad \mu^{+}_j(a(\beta)) < -\zeta^{+}_{a(\beta),\beta}\}$$

by using Propositions 2.3 and 2.4

$$\supset \{j \in \mathbb{N}; \quad \mu_j + C_1 a(\beta) j^2 < \tfrac{1}{4}\beta^2 - 2\beta^2 \exp(-\tfrac{1}{2}\beta a(\beta))\}$$

since $\mu_j \le [j/2]^2(\pi/L)^2 + (1/4)\gamma_+^2$

$$\supset \left\{ j \in \mathbb{N};\quad 4\left[\frac{j}{2}\right]^2 \left(\frac{\pi}{L}\right)^2 + C_1(\beta^{-1}\log\beta)j^2 < \frac{1}{4}\beta^2 - \frac{2}{\beta} - \frac{1}{4}\gamma_+^2 \right\}$$

since $\beta \ge 2$

$$\supset \left\{ j \in \mathbb{N};\quad j^2 \left(\frac{\pi}{L}\right)^2 + C_1(\beta^{-1}\log\beta)j^2 < \frac{1}{4}\beta^2 - 1 - \frac{1}{4}\gamma_+^2 \right\}$$
$$\supset \left\{ j \in \mathbb{N};\quad j^2 \left(\frac{\pi}{L}\right)^2 + C_1(\beta^{-1}\log\beta)j^2 \le \frac{1}{4}(\beta - C_2)^2 \right\}$$
$$= \left\{ j \in \mathbb{N};\quad j \le \frac{1}{2}(\beta - C_2)\left(\left(\frac{\pi}{L}\right)^2 + C_1\beta^{-1}\log\beta\right)^{-1/2} \right\}.$$

Furthermore, from

$$\frac{1}{2}(\beta - C_2)\left(\left(\frac{\pi}{L}\right)^2 + C_1\beta^{-1}\log\beta\right)^{-1/2} = \frac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta \to \infty,$$

we infer that

$$\#K_\beta^+ \ge \frac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta \to \infty. \tag{3.2}$$

Similarly we get

$$K_\beta^- = \{ j \in \mathbb{N};\quad \mu_j^-(a(\beta)) < -\zeta_{a(\beta),\beta}^- \}$$
$$\subset \left\{ j \in \mathbb{N};\quad \mu_j - C_1 a(\beta) j^2 < \frac{1}{4}\beta^2 + \frac{2205}{4\beta} \right\}$$

since $2(j-1) \ge j$ for $j \ge 2$

$$\subset \{1\} \cup \left\{ j \ge 2;\ (j-1)^2\left(\frac{\pi}{L}\right)^2 - 4C_1(\beta^{-1}\log\beta)(j-1)^2 < \frac{1}{4}\beta^2 + \frac{2205}{4\beta} + \frac{1}{4}\gamma_+^2 \right\}$$
$$= \{1\} \cup \left\{ j \ge 2;\quad j < 1 + \left(\frac{1}{4}\beta^2 + \frac{2205}{4\beta} + \frac{1}{4}\gamma_+^2\right)^{1/2} \times \left(\left(\frac{\pi}{L}\right)^2 - 4C_1\beta^{-1}\log\beta\right)^{-1/2} \right\}.$$

However

$$1 + \left(\frac{1}{4}\beta^2 + \frac{2205}{4\beta} + \frac{1}{4}\gamma_+^2\right)^{1/2}\left(\left(\frac{\pi}{L}\right)^2 - 4C_1\beta^{-1}\log\beta\right)^{-1/2} = \frac{L\beta}{2\pi} + \mathcal{O}(\log\beta)$$

as $\beta \to \infty$, which leads to

$$\#K_\beta^- \le \frac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta \to \infty. \tag{3.3}$$

Since $\tau^-_{\beta,j} < \tau^+_{\beta,j}$, we get $K^-_\beta \supset K^+_\beta$. Combining this with (3.2) and (3.3), we get the assertion of Proposition 3.1. □

We also need the following result to estimate the second eigenvalue of $T^-_{a,\beta}$.

Proposition 3.2. *Let* $0 < a < 1/\sqrt{2}\gamma_+$ *and* $\beta > 0$. *Then* $T^-_{a,\beta}$ *has no eigenvalue in* $[0, \min\{\pi^2/16a^2, \beta\gamma_+/2, \beta^2\})$.

Proof. Let $k > 0$. As in the proof of Proposition 2.4, we infer that k^2 is an eigenvalue of $T^-_{a,\beta}$ if and only if k solves either

$$\tan ka = \frac{k}{\gamma_+} \tag{3.4}$$

or

$$\tan ka = \frac{\beta + 2k\gamma_+}{\beta\gamma_+ - 2k^2}\beta. \tag{3.5}$$

For $k \in (0, \pi/4a)$, we have

$$\tan ka < \sqrt{2}\,\sin ka < \sqrt{2}ka < \frac{k}{\gamma_+}. \tag{3.6}$$

Thus (3.4) has no solution in $(0, \pi/4a)$. For $k \in (0, \min\{\pi/4a, \sqrt{\beta\gamma_+}/\sqrt{2}, \beta\})$, we have

$$\frac{\beta + 2k\gamma_+}{\beta\gamma_+ - 2k^2}\beta - \frac{k}{\gamma_+} = \frac{\beta\gamma_+(\beta - k) + 2k(\gamma_+)^2\beta + 2k^3}{(\beta\gamma_+ - 2k^2)\gamma_+} > 0.$$

This together with (3.6) implies that (3.5) has no solution in $(0, \min\{\pi/4a, \sqrt{\beta\gamma_+}/\sqrt{2}, \beta\})$. Consequently, $T^-_{a,\beta}$ has no eigenvalue in $(0, \min\{\pi^2/16a^2, \beta\gamma_+/2, \beta^2\})$.

Next we show that 0 is not an eigenvalue of $T^-_{a,\beta}$. As in the proof of Proposition 2.4, we infer that 0 is an eigenvalue of $T^-_{a,\beta}$ if and only if either $\gamma_+ a = 1$ or $\beta(\gamma_+ a - 1) = 2\gamma_+$ holds. Since $0 < a < 1/\sqrt{2}\gamma_+$ and $\beta > 0$, we have $\gamma_+ a < 1$ and $\beta(\gamma_+ a - 1) < 2\gamma_+$. Hence 0 is not an eigenvalue of $T^-_{a,\beta}$, and the proof is complete. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Let us first show that

$$N^-(\tilde{H}^-_{a(\beta),\beta}) = \#K^-_\beta \quad \text{for sufficiently large } \beta > 0. \tag{3.7}$$

Recall that $\{\xi^-_{\beta,j} + \mu^-_k(a(\beta))\}_{j,k\in\mathbb{N}}$ is a sequence of all eigenvalues of $\tilde{H}^-_{a(\beta),\beta}$ counted with multiplicity. From Proposition 3.2, we have

$$\xi^-_{\beta,2} \geq \min\left\{\frac{\pi^2}{16a(\beta)^2}, \frac{\beta\gamma_+}{2}, \beta^2\right\}.$$

This together with (2.10) implies that there exists $\beta_0 > 0$ such that $\xi^-_{\beta,2} + \mu^-_1(a(\beta)) > 0$ for $\beta \geq \beta_0$. We obtain

$$\xi^-_{\beta,j} + \mu^-_k(a(\beta)) > 0 \quad \text{for} \quad j \geq 2, \quad k \geq 1, \quad \text{and} \quad \beta \geq \beta_0.$$

Thus we get

$$N^-(\tilde{H}^-_{a(\beta),\beta}) = \#\{(j,k) \in \mathbb{N}^2; \quad \xi^-_{\beta,j} + \mu^-_k(a(\beta)) < 0\}$$
$$= \#\{j \in \mathbb{N}; \quad \tau^-_{\beta,j} < 0\} = \#K^-_\beta \quad \text{for} \quad \beta \geq \beta_0.$$

In this way we obtain (3.7). From (3.1), we get

$$\#K^+_\beta \leq \#\sigma_{\rm d}(H_\beta) \leq N^-(\tilde{H}^-_{a(\beta),\beta}).$$

This together with (3.7) and Proposition 3.1implies the assertion of Theorem 2. □

Remark 3.3. We can also prove (1.5) in the case that γ is an open C^4 Jordan curve. Indeed, it suffices to use the following operators $\hat{H}^\pm_{a,\beta}$ instead of $\tilde{H}^\pm_{a,\beta} = U^\pm_a \otimes 1 + 1 \otimes T^\pm_{a,\beta}$:

$$\hat{H}^\pm_{a,\beta} := \hat{U}^\pm_a \otimes 1 + 1 \otimes T^\pm_{a,\beta} \quad \text{in} \quad L^2(0,L) \otimes L^2(-a,a) = L^2((0,L) \times (-a,a)),$$

$$\hat{U}^+_a := -(1 - a\gamma_+)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_+(s) \quad \text{in} \quad L^2(0,L)$$

with the Dirichlet boundary condition,

$$\hat{U}^-_a := -(1 + a\gamma_+)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_-(s) \quad \text{in} \quad L^2(0,L)$$

with the Neumann boundary condition.

Remark 3.4. The operator H_β can be defined in a different way via a boundary condition on Γ. Let $n(x)$ be the outward normal vector field on Γ. In [3] Remark 4.1, it is shown that the set

$$\left\{ f \in H^1(\mathbb{R}^2) \cap C_0(\mathbb{R}^2); \quad f|_{\mathbb{R}^2\setminus\Gamma} \in H^2(\mathbb{R}^2\setminus\Gamma) \cap C^\infty(\mathbb{R}^2\setminus\Gamma), \right.$$
$$\left. \frac{\partial f}{\partial n_+}(x) - \frac{\partial f}{\partial n_-}(x) = -\beta f(x), \quad \text{for} \quad x \in \Gamma \right\}$$

is the core of H_β, where $\partial f/\partial n_+(x)$ and $\partial f/\partial n_-(x)$ are the derivatives in the direction of $n(x)$ and $-n(x)$, respectively, at the point x.

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